Counterexamples in Analysis

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Abstract

The counterexamples are used for better comprehension of underlying concept in a theorem or definition. This paper is about the counterexamples in Mathematical Analysis, that we construct them to be apparently correct statements by a few change in the conditions of a valid proposition. Our aim in this paper is to present how the counterexamples help to boost the level of understanding of mathematical Analysis concepts.

Keywords : Counterexamples, Topological space ,Complete metric space, topology, Convergence or divergence of Sequences and Infinite Series.

Introduction

Examples are very important in mathematics such that guide mathematician to new idea. In this between, there are examples that we call them counterexamples, that with their help we receive deeper conceptual understanding and enhance critical thinking skills. There are mathematic counterexamples in different books, and in this article we try with change condition some theorem or definition in mathematic analysis make mistake proposition and present counterexamples for everyone until high light them.

Apparently Correct Proposition

Proposition1. The intersection of every decreasing sequence of nonempty closed and bounded sets is an empty.

Counterexample In fact this proposition says, we can change compact condition whit closed and bounded condition, in following theorem:

Theorem: If $\{k_n\}$ decreasing sequence of nonempty compact sets then $\bigcap_{i=1}^{\infty} k_n$ is an empty.

But compact is equal to closed and bounded in complete space. Then we can to search counterexample in an incomplete space, for instance following example: In the space $\Re$ with the bounded metric $d(x,y) = \frac{|x-y|}{1+|x-y|}$,let $F_n = [n, \infty)$, n =1,2, $\ldots$. Then each F_n is closed and bounded, and $\bigcap_{i=1}^{\infty} F_n$.[4]

Proposition2.Completeness is a topological property.

Counterexample Let the space ($\aleph$,d) of natural numbers with the metric $d(m,n) = \frac{|m-n|}{mn}$, and the natural numbers with the standard metric. The natural numbers set has the discrete topology since every one-point set is open, but the sequence $\{n\}$ is a nonconvergent Cauchy sequence.

This example demonstrates that completeness is not a topological property, since the space $\aleph$ with the standard metric is both complete and discrete. In other words, it is possible for two metric spaces to be homeomorphic even though one is complete and one is not. Another example of two such space consists of the homeomorphic intervals $(-\infty, +\infty)$ and $(0,1)$ of which only the first is complete in the standard metric of $\Re$.[4]

Proposition3. The intersection of every decreasing sequence of nonempty closed balls

in a complete metric space is a nonempty.

Counterexample In the space ($\aleph$,d) of natural numbers with the metric,

$$d(m,n) = \begin{cases} 0 & m = n \\ (1 + \frac{1}{m+n}) & m \neq n \end{cases}$$

let

$$B_n = \{m | d(m,n) \leq (1 + \frac{1}{2n})\} = \{m | (1 + \frac{1}{m+n}) \leq (1 + \frac{1}{2n})\} = \{m | m \geq n\} = \{n, n+1, \ldots\}$$

for $n = 1, 2, \ldots$. Then $\{B_n\}$ satisfies the stipulated conditions, and the space is complete since every Cauchy sequence is "ultimately constant."

This counterexample demonstrate that we have to pay attention to the condition of sets being open in Baire's category theorem.(see more example in [1],[2],[3],[4],[5])

Proposition4. In topological space, limits of sequences are unique.

Counterexample We know that the limits of sequences in Hausdorff spaces are unique. We make two counterexamples. First counterexample: Let X be an infinite set, and let τ consist of ϕ and the complements of all finite subsets of$X$. Then every sequence of distinct points of X converges to every member of X. Second counterexample: Any space with the trivial topology and consisting of more than one point has this property since in this space every sequence converges to every point.

Proposition5. If a mapping from one topological space onto another is continuous then it either is open or closed.

Counterexample First counterexample:Let X is an arbitrary space,$Y = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, Y\}$ is a topology in Y. Now the constant function $f : X \to Y, f(x) = a$ which is continuous neither is open nor closed. Second counterexample: Let X be the space $\Re$ with the discrete topology, let Y be the space $\Re$ with the standard topology, and let the mapping be the identity mapping.

Proposition6. If a mapping from one topological space onto another is continuous and open then is a closed mapping too.

Counterexample First counterexample: Let$X$ is a arbitrary space,$Y = \{a, b, c\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, Y\}$ is topology in Y. Now constant function $f : X \to Y, f(x) = b$ that is continuous and open but isn't closed.

Second counterexample:Be $\Re^2$ with the standard topology and be $\Re$ with the usual topology, let projection $\pi_1 : \Re \times \Re \to \Re, \pi_1(x,y) = x$ then π_1 is a continuous and open and is not closed (because $F = \{(x,y)|xy = 1\}$ is closed in $\Re^2$) but $\pi_1(F) = \Re - \{0\}$ is not closed in $\Re$.

Proposition7. If mapping of one topological space onto another is open then is closed and continuous mapping.

Counterexample Let (X,τ) be the plane with τ consisting of and complements of countable sets, let (Y,τ') be $\Re$ with τ' consisting of ϕ and complements of finite sets, and let the mapping be the projection $\pi_1 : X \to Y, \pi_1(x,y) = x$. Then π_1 is open since any nonempty open set in (X,τ) must contain some horizontal line, whose image is $\Re$. On the other hand, π_1 is not closed since the set of points(n,0), where $n \in \aleph$, is closed in (X,τ), but its image is not closed in (Y,τ'), and since the inverse image of any open set in (Y,τ') that is a proper subset of Y cannot be open in (X,τ), π_1 is not continuous.

Proposition8. If mapping of one topological space onto another is closed and continuous then is open mapping too.

Counterexample First counterexample: Let X is a arbitrary space, $Y = \{a,b,c\}$ and $\tau = \{\phi, \{b\}, \{a,b\}, Y\}$ is topology in Y. Now constant function $f : X \to Y, f(x) = c$ that is continuous and closed but is not open. Second counterexample: Let X and Y be the closed interval [0,2] with the usual topology, and let;

$$f : X \to Y$$

$$f(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ (x-1) & 1 \leq x \leq 2 \end{cases}$$

Then f is clearly continuous, and hence closed since X and Y are compact metric space. On the other hand, $f((0,1))$ is not open in Y.

Proposition9. If $\{|a_n|\}$ is a convergent sequence, then $\{a_n\}$ is a convergent sequence too.

Counterexample Let $\{|(-1)^n|\}$ that is a convergent sequence, but $\{(-1)^n\}$ is a divergent sequence.

Proposition10. If$\{a_n\}$ is a divergent sequence, then it has infinite dimension.

Counterexample First counterexample: $a_n = i^n$, $(i^2 = -1)$ is a divergent sequence that

it has finite dimension. Second counterexample: Alternating sequence $a_n = (-1)^n$ is a divergent sequence but it has finite dimension.

Proposition11. If $\{a_n\}$ is a divergent sequence and its dimension is infinite(with infinite dimension), then is not bounded.

Counterexample $\{\sin n\}$ is a divergent sequence with infinite dimension but is a bounded sequence.

Proposition12. Every Cauchy sequence is a convergent.

Counterexample Let$\{\frac{1}{n}\}$ be a sequence on (0,1], then $\{\frac{1}{n}\}$ is a divergent sequence on (0,1], but $\{\frac{1}{n}\}$ is a Cauchy sequence on (0,1]. In fact proposition established for complete metric space.

Proposition13. If $b_n \leq a_n$ for all n=1,2,3,... and $\sum a_n$ is a convergent then $\sum b_n$ is a convergent too.

Counterexample $b_n = \frac{(-1)}{n} \leq 0 = a_n$, for all n=1,2,3,..., and $\sum a_n$ is a convergent to zero but $\sum b_n = \sum \frac{(-1)}{n}$ is a divergent.

Proposition14. If $|b_n| \leq |a_n|$ for all n=1,2,3,... and $\sum a_n$ is a convergent then $\sum b_n$ is a convergent too.

Counterexample Let $a_n = \frac{(-1)^n}{n}$,$b_n = \frac{1}{n}$ then $|b_n| \leq |a_n|$ for all n=1,2,3,... and we know that alternating series $\sum a_n = \sum \frac{(-1)^n}{n}$ is a convergent but harmonic series $\sum a_n = \sum \frac{1}{n}$ is a divergent.

Proposition15. Cauchy product two divergent series is a divergent.

Counterexample The Cauchy product series of the two series;

$$2 + \sum_{i=1}^{+\infty} 2^n = 2 + 2 + 2^2 + 2^3 + \ldots,$$

$$-1 + \sum_{i=1}^{+\infty} 1 = -1 + 1 + 1 + 1 + \ldots$$

is

$$-2 + 0 + 0 + 0 + \ldots$$

More generally, if $a_n = a^n$ for $n \geq 1$ and if $b_n = b^n$ for $n \geq 1$, and if $a \neq b$, the term c_n of the product series of $\sum a_n$ and $\sum b_n$, for $n \geq 1$, is

$$c_n = a_0 b^n + b_0 a^n + a^{(n-1)} b + a^{(n-2)} b^2 + \ldots + a b^{(n-1)} = a_0 b^n + b_0 a^n - a^n - b^n + \frac{a^{n+1} - b^{n+1}}{a - b}$$

$$= \frac{a^n(a+(b_0-1)(a-b)) - b^n(b+(1-a_0)(a-b))}{a-b}$$

and therefore $c_n = 0$ in case $a = (1-b_0)(a-b)$ and $b = (a_0-1)(a-b)$. If a and b chosen so that $a - b = 1$, then a_0 and b_0 are given by $a_0 = b+1 = a, b_0 = 1-a = -b$.[4]

Proposition16. If $f(x)$ for all $x \geq 1$ is a continuous and $\int_1^{+\infty} f(x)dx$ is a convergent, then $\sum_{n=1}^{+\infty} f(n)$ is a convergent too.

Counterexample Let continuous function $f(x) = \cos x^2$ on $[1, +\infty)$ then $\int_1^{+\infty} f(x)dx$ is a convergent but $\sum_{n=1}^{+\infty} \cos n^2$ is a divergent.

Proposition17. If $f(x)$ is a positive function and a.e. continuous on $[1, +\infty)$ and $\int_1^{+\infty} f(x)dx$ is a convergent then $\sum_{n=1}^{+\infty} f(n)$ is a convergent too.

Counterexample Let$g(x)$ on $[1, +\infty)$

$$g(x) = \begin{cases} 0 & x \in [1, +\infty) - Z \\ 1 & x \in Z \bigcap [1, +\infty) \end{cases}$$

now let $f(x) = g(x) + \frac{1}{x^2}$ therefore $f(x)$ for each $x \geq 1$ is a positive and a.e. continuous function and $\int_1^{+\infty} f(x)dx = \int_1^{+\infty} \frac{1}{x^2}dx$ is a convergent but $\sum_{n=1}^{+\infty} f(n)$ is a divergent.

Proposition18. If a function $f(x)$ is continuous and $\int_a^{+\infty} f(x)dx$ converges then $\lim_{x\to\infty} f(x) = 0$.

Counterexample The Fresnel integral $\int_0^{+\infty} \sin x^2 dx$ converges but $\lim_{x\to\infty} \sin x^2$ does not exist.[6]

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